

# Gauge and Einstein Gravity from Non-Abelian Gauge Models on Noncommutative Spaces

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## Abstract

Following the formalism of enveloping algebras and star product calculus we formulate and analyze a model of gauge gravity on noncommutative spaces and examine the conditions of its equivalence to general relativity. The corresponding Seiber–Witten maps are established which allow the definition of respective dynamics for a finite number of gravitational gauge field components on noncommutative spaces.

**Keywords:** noncommutative geometry, gauge gravity, general relativity

## 1 Introduction

In the last years much work has been made in noncommutative extensions of physical theories. It was not possible to formulate gauge theories on noncommutative spaces [1, 2, 3, 4] with Lie algebra valued infinitesimal transformations and with Lie algebra valued gauge fields. In order to avoid the problem the authors of [5] suggested to use enveloping algebras of the Lie algebras for setting this type of gauge theories and showed that in spite of the fact that such enveloping algebras are infinite-dimensional one can restrict them in a way that it would be a dependence on the Lie algebra valued parameters and the Lie algebra valued gauge fields and their spacetime derivatives only.

A still presented drawback of noncommutative geometry and physics is that there is not yet formulated a generally accepted approach to interactions of elementary particles coupled to gravity. There are improved Connes–Lott and

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Chamsedine–Connes models of noncommutative geometry [6] which yielded action functionals tying together the gravitational and Yang–Mills interactions and gauge bosons the Higgs sector (see also the approaches [7] and, for an outline of recent results, [8]).

In this paper we follow the method of restricted enveloping algebras [3, 5] and construct gauge gravitational theories by stating corresponding structures with semisimple or nonsemisimple Lie algebras and their extensions. We consider power series of generators for the affine and non linear realized de Sitter gauge groups and compute the coefficient functions of all the higher powers of the generators of the gauge group which are functions of the coefficients of the first power. Such constructions are based on the Seiberg–Witten map [2] and on the formalism of  $\ast$ -product formulation of the algebra [9] when for functional objects, being functions of commuting variables, there are associated some algebraic noncommutative properties encoded in the  $\ast$ -product.

The concept of gauge theory on noncommutative spaces was introduced in a geometric manner [4] by defining the covariant coordinates without speaking about derivatives and this formalism was developed for quantum planes [10]. In this paper we shall prove the existence for noncommutative spaces of gauge models of gravity which agrees with usual gauge gravity theories being equivalent or extending the general relativity theory (see works [11, 12] for locally isotropic spaces and corresponding reformulations and generalizations respectively for anholonomic frames [13] and locally anisotropic (super) spaces [14]) in the limit of commuting spaces.

## 2 $\ast$ -Products and Enveloping Algebras in Noncommutative Spaces

For a noncommutative space the coordinates  $\hat{u}^i$ , ( $i = 1, \dots, N$ ) satisfy some noncommutative relations of type

$$[\hat{u}^i, \hat{u}^j] = \begin{cases} i\theta^{ij}, & \theta^{ij} \in \mathbb{C}, \text{ canonical structure;} \\ i f_k^{ij} \hat{u}^k, & f_k^{ij} \in \mathbb{C}, \text{ Lie structure;} \\ i C_{kl}^{ij} \hat{u}^k \hat{u}^l, & C_{kl}^{ij} \in \mathbb{C}, \text{ quantum plane structure} \end{cases} \quad (1)$$

where  $\mathbb{C}$  denotes the complex number field.

The noncommutative space is modeled as the associative algebra of  $\mathbb{C}$ ; this algebra is freely generated by the coordinates modulo ideal  $\mathcal{R}$  generated by the relations (one accepts formal power series)  $\mathcal{A}_u = \mathbb{C}[[\hat{u}^1, \dots, \hat{u}^N]]/\mathcal{R}$ . One restricts attention [5] to algebras having the (so-called, Poincaré–Birkhoff–Witt) property that any element of  $\mathcal{A}_u$  is defined by its coefficient function and vice versa,

$$\hat{f} = \sum_{L=0}^{\infty} f_{i_1, \dots, i_L} : \hat{u}^{i_1} \dots \hat{u}^{i_L} : \quad \text{when } \hat{f} \sim \{f_i\},$$

where  $: \hat{u}^{i_1} \dots \hat{u}^{i_L} :$  denotes that the basis elements satisfy some prescribed order (for instance, the normal order  $i_1 \leq i_2 \leq \dots \leq i_L$ , or, another example,

are totally symmetric). The algebraic properties are all encoded in the so-called diamond ( $\diamond$ ) product which is defined by

$$\widehat{f}\widehat{g} = \widehat{h} \sim \{f_i\} \diamond \{g_i\} = \{h_i\}.$$

In the mentioned approach to every function  $f(u) = f(u^1, \dots, u^N)$  of commuting variables  $u^1, \dots, u^N$  one associates an element of algebra  $\widehat{f}$  when the commuting variables are substituted by anticommuting ones,

$$f(u) = \sum f_{i_1 \dots i_L} u^1 \dots u^N \rightarrow \widehat{f} = \sum_{L=0}^{\infty} f_{i_1, \dots, i_L} : \widehat{u}^{i_1} \dots \widehat{u}^{i_L} :$$

when the  $\diamond$ -product leads to a bilinear  $*$ -product of functions (see details in [4])

$$\{f_i\} \diamond \{g_i\} = \{h_i\} \sim (f * g)(u) = h(u).$$

The  $*$ -product is defined respectively for the cases (1)

$$f * g = \begin{cases} \exp[\frac{i}{2} \frac{\partial}{\partial u^i} \theta^{ij} \frac{\partial}{\partial u'^j}] f(u) g(u')|_{u' \rightarrow u}, & \text{canonical structure;} \\ \exp[\frac{i}{2} u^k g_k(i \frac{\partial}{\partial u'}, i \frac{\partial}{\partial u''})] f(u') g(u'')|_{u' \rightarrow u, u'' \rightarrow u}, & \text{Lie structure;} \\ q^{\frac{1}{2}(-u' \frac{\partial}{\partial u'} v \frac{\partial}{\partial v} + u \frac{\partial}{\partial u} v' \frac{\partial}{\partial v'})} f(u, v) g(u', v')|_{v' \rightarrow v, v \rightarrow u}, & \text{quantum plane,} \end{cases}$$

where there are considered values of type

$$\begin{aligned} e^{ik_n \widehat{u}^n} e^{ip_n \widehat{u}^n} &= e^{i\{k_n + p_n + \frac{1}{2} g_n(k, p)\} \widehat{u}^n}, \\ g_n(k, p) &= -k_i p_j f_n^{ij} + \frac{1}{6} k_i p_j (p_k - k_k) f_m^{ij} f_n^{mk} + \dots, \\ e^A e^B &= e^{A+B + \frac{1}{2}[A, B] + \frac{1}{12}([A, [A, B]] + [B, [B, A]])} + \dots \end{aligned} \quad (2)$$

and for the coordinates on quantum (Manin) planes one holds the relation  $uv = qvu$ .

A non-abelian gauge theory on a noncommutative space is given by two algebraic structures, the algebra  $\mathcal{A}_u$  and a non-abelian Lie algebra  $\mathcal{A}_I$  of the gauge group with generators  $I^1, \dots, I^S$  and the relations

$$[I^s, I^p] = i f_{\frac{sp}{t}}^t I^t. \quad (3)$$

In this case both algebras are treated on the same footing and one denotes the generating elements of the big algebra by  $\widehat{u}^i$ ,

$$\begin{aligned} \widehat{z}^i &= \{\widehat{u}^1, \dots, \widehat{u}^N, I^1, \dots, I^S\}, \\ \mathcal{A}_z &= \mathbb{C}[[\widehat{u}^1, \dots, \widehat{u}^{N+S}]]/\mathcal{R}, \end{aligned}$$

and the  $*$ -product formalism is to be applied for the whole algebra  $\mathcal{A}_z$  when there are considered functions of the commuting variables  $u^i$  ( $i, j, k, \dots = 1, \dots, N$ ) and  $I^s$  ( $s, p, \dots = 1, \dots, S$ ).

For instance, in the case of a canonical structure for the space variables  $u^i$  we have

$$(F * G)(u) = e^{\frac{i}{2}(\theta^{ij} \frac{\partial}{\partial u^i} \frac{\partial}{\partial u'^j} + t^s g_s(i \frac{\partial}{\partial t'}, i \frac{\partial}{\partial t''}))} F(u', t') G(u'', t'')|_{t' \rightarrow t, t'' \rightarrow t}^{u' \rightarrow u, u'' \rightarrow u}. \quad (4)$$

This formalism was developed in [5] for general Lie algebras. In this paper we shall consider those cases when in the commuting limit one obtains the gauge gravity and general relativity theories.

### 3 Enveloping Algebras for Gravitational Gauge Connections

To define gauge gravity theories on noncommutative space we first introduce gauge fields as elements the algebra  $\mathcal{A}_u$  that form representation of the generator  $I$ -algebra for the de Sitter gauge group. For commutative spaces it is known [11, 12, 14] that an equivalent reexpression of the Einstein theory as a gauge like theory implies, for both locally isotropic and anisotropic spacetimes, the nonsemisimplicity of the gauge group, which leads to a nonvariational theory in the total space of the bundle of locally adapted affine frames (to this class one belong the gauge Poincare theories; on metric-affine and gauge gravity models see original results and reviews in [15]). By using auxiliary bilinear forms, instead of degenerated Killing form for the affine structural group, on fiber spaces, the gauge models of gravity can be formulated to be variational. After projection on the base spacetime, for the so-called Cartan connection form, the Yang-Mills equations transforms equivalently into the Einstein equations for general relativity [11]. A variational gauge gravitational theory can be also formulated by using a minimal extension of the affine structural group  $\mathcal{A}f_{3+1}(\mathbb{R})$  to the de Sitter gauge group  $S_{10} = SO(4+1)$  acting on  $\mathbb{R}^{4+1}$  space.

#### 3.1 Nonlinear gauge theories of de Sitter group in commutative spaces

Let us consider the de Sitter space  $\Sigma^4$  as a hypersurface given by the equations  $\eta_{AB}u^Au^B = -l^2$  in the four dimensional flat space enabled with diagonal metric  $\eta_{AB}, \eta_{AA} = \pm 1$  (in this section  $A, B, C, \dots = 1, 2, \dots, 5$ ), where  $\{u^A\}$  are global Cartesian coordinates in  $\mathbb{R}^5$ ;  $l > 0$  is the curvature of de Sitter space. The de Sitter group  $S_{(\eta)} = SO_{(\eta)}(5)$  is defined as the isometry group of  $\Sigma^5$ -space with 6 generators of Lie algebra  $so_{(\eta)}(5)$  satisfying the commutation relations

$$[M_{AB}, M_{CD}] = \eta_{AC}M_{BD} - \eta_{BC}M_{AD} - \eta_{AD}M_{BC} + \eta_{BD}M_{AC}. \quad (5)$$

Decomposing indices  $A, B, \dots$  as  $A = (\underline{\alpha}, 5), B = (\underline{\beta}, 5), \dots$ , the metric  $\eta_{AB}$  as  $\eta_{AB} = (\eta_{\underline{\alpha}\underline{\beta}}, \eta_{55})$ , and operators  $M_{AB}$  as  $M_{\underline{\alpha}\underline{\beta}} = \mathcal{F}_{\underline{\alpha}\underline{\beta}}$  and  $P_{\underline{\alpha}} = l^{-1}M_{5\underline{\alpha}}$ , we can write (5) as

$$\begin{aligned} [\mathcal{F}_{\underline{\alpha}\underline{\beta}}, \mathcal{F}_{\underline{\gamma}\underline{\delta}}] &= \eta_{\underline{\alpha}\underline{\gamma}}\mathcal{F}_{\underline{\beta}\underline{\delta}} - \eta_{\underline{\beta}\underline{\gamma}}\mathcal{F}_{\underline{\alpha}\underline{\delta}} + \eta_{\underline{\beta}\underline{\delta}}\mathcal{F}_{\underline{\alpha}\underline{\gamma}} - \eta_{\underline{\alpha}\underline{\delta}}\mathcal{F}_{\underline{\beta}\underline{\gamma}}, \\ [P_{\underline{\alpha}}, P_{\underline{\beta}}] &= -l^{-2}\mathcal{F}_{\underline{\alpha}\underline{\beta}}, \quad [P_{\underline{\alpha}}, \mathcal{F}_{\underline{\beta}\underline{\gamma}}] = \eta_{\underline{\alpha}\underline{\beta}}P_{\underline{\gamma}} - \eta_{\underline{\alpha}\underline{\gamma}}P_{\underline{\beta}}, \end{aligned} \quad (6)$$

where we have indicated the possibility to decompose the Lie algebra  $so_{(\eta)}(5)$  into a direct sum,  $so_{(\eta)}(5) = so_{(\eta)}(4) \oplus V_4$ , where  $V_4$  is the vector space stretched on vectors  $P_{\underline{\alpha}}$ . We remark that  $\Sigma^4 = S_{(\eta)}/L_{(\eta)}$ , where  $L_{(\eta)} = SO_{(\eta)}(4)$ . For  $\eta_{AB} = \text{diag}(1, -1, -1, -1)$  and  $S_{10} = SO(1, 4)$ ,  $L_6 = SO(1, 3)$  is the group of Lorentz rotations.

In this paper the generators  $I^a$  and structure constants  $f_{\underline{t}}^{sp}$  from (3) are parametrized just to obtain de Sitter generators and commutations (6).

The action of the group  $S_{(\eta)}$  can be realized by using  $4 \times 4$  matrices with a parametrization distinguishing subgroup  $L_{(\eta)}$  :

$$B = bB_L, \quad (7)$$

where

$$B_L = \begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix},$$

$L \in L_{(\eta)}$  is the de Sitter boost matrix transforming the vector  $(0, 0, \dots, \rho) \in \mathbb{R}^5$  into the arbitrary point  $(V^1, V^2, \dots, V^5) \in \Sigma_\rho^5 \subset \mathcal{R}^5$  with curvature  $\rho$ ,  $(V_A V^A = -\rho^2, V^A = t^A \rho)$ . Matrix  $b$  can be expressed as

$$b = \begin{pmatrix} \delta^\alpha_{\underline{\beta}} + \frac{t^\alpha t_{\underline{\beta}}}{(1+t^5)} & t^\alpha \\ t_{\underline{\beta}} & t^5 \end{pmatrix}.$$

The de Sitter gauge field is associated with a  $so_{(\eta)}(5)$ -valued connection 1-form

$$\tilde{\Omega} = \begin{pmatrix} \omega^\alpha_{\underline{\beta}} & \tilde{\theta}^\alpha \\ \tilde{\theta}_{\underline{\beta}} & 0 \end{pmatrix}, \quad (8)$$

where  $\omega^\alpha_{\underline{\beta}} \in so(4)_{(\eta)}$ ,  $\tilde{\theta}^\alpha \in \mathcal{R}^4$ ,  $\tilde{\theta}_{\underline{\beta}} \in \eta_{\underline{\beta}\alpha} \tilde{\theta}^\alpha$ .

Because  $S_{(\eta)}$ -transforms mix  $\omega^\alpha_{\underline{\beta}}$  and  $\tilde{\theta}^\alpha$  fields in (8) (the introduced parametrization is invariant on action on  $SO_{(\eta)}(4)$  group we cannot identify  $\omega^\alpha_{\underline{\beta}}$  and  $\tilde{\theta}^\alpha$ , respectively, with the connection  $\Gamma^\alpha_{\beta\gamma}$  and the fundamental form  $\chi^\alpha$  in a metric-affine spacetime. To avoid this difficulty we consider [12] a nonlinear gauge realization of the de Sitter group  $S_{(\eta)}$ , namely, we introduce into consideration the nonlinear gauge field

$$\Gamma = b^{-1} \tilde{\Omega} b + b^{-1} db = \begin{pmatrix} \Gamma^\alpha_{\underline{\beta}} & \theta^\alpha \\ \theta_{\underline{\beta}} & 0 \end{pmatrix}, \quad (9)$$

where

$$\begin{aligned} \Gamma^\alpha_{\underline{\beta}} &= \omega^\alpha_{\underline{\beta}} - (t^\alpha D t_{\underline{\beta}} - t_{\underline{\beta}} D t^\alpha) / (1 + t^5), \\ \theta^\alpha &= t^5 \tilde{\theta}^\alpha + D t^\alpha - t^\alpha (d t^5 + \tilde{\theta}_{\underline{\gamma}} t^{\underline{\gamma}}) / (1 + t^5), \\ D t^\alpha &= d t^\alpha + \omega^\alpha_{\underline{\beta}} t^{\underline{\beta}}. \end{aligned}$$

The action of the group  $S(\eta)$  is nonlinear, yielding transforms

$$\Gamma' = L' \Gamma (L')^{-1} + L' d(L')^{-1}, \theta' = L \theta,$$

where the nonlinear matrix-valued function  $L' = L'(t^\alpha, b, B_T)$  is defined from  $B_b = b' B_{L'}$  (see the parametrization (7)). The de Sitter algebra with generators (6) and nonlinear gauge transforms of type (9) is denoted  $\mathcal{A}_I^{(dS)}$ .

### 3.2 Enveloping nonlinear de Sitter algebra valued connection

Let now consider a noncommutative space. In this case the gauge fields are elements of the algebra  $\hat{\psi} \in \mathcal{A}_I^{(dS)}$  that form the nonlinear representation of the de Sitter algebra  $so_{(\eta)}$  (5) when the whole algebra is denoted  $\mathcal{A}_z^{(dS)}$ . Under a nonlinear de Sitter transformation the elements transform as follows

$$\delta\hat{\psi} = i\hat{\gamma}\hat{\psi}, \hat{\psi} \in \mathcal{A}_u, \hat{\gamma} \in \mathcal{A}_z^{(dS)},$$

So, the action of the generators (6) on  $\hat{\psi}$  is defined as this element is supposed to form a nonlinear representation of  $\mathcal{A}_I^{(dS)}$  and, in consequence,  $\delta\hat{\psi} \in \mathcal{A}_u$  despite  $\hat{\gamma} \in \mathcal{A}_z^{(dS)}$ . It should be emphasized that independent of a representation the object  $\hat{\gamma}$  takes values in enveloping de Sitter algebra and not in a Lie algebra as would be for commuting spaces. The same holds for the connections that we introduce (similarly to [4]) in order to define covariant coordinates

$$\hat{U}^\nu = \hat{u}^\nu + \hat{\Gamma}^\nu, \hat{\Gamma}^\nu \in \mathcal{A}_z^{(dS)}.$$

The values  $\hat{U}^\nu\hat{\psi}$  transforms covariantly,  $\delta\hat{U}^\nu\hat{\psi} = i\hat{\gamma}\hat{U}^\nu\hat{\psi}$ , if and only if the connection  $\hat{\Gamma}^\nu$  satisfies the transformation law of the enveloping nonlinear realized de Sitter algebra,

$$\delta\hat{\Gamma}^\nu\hat{\psi} = -i[\hat{u}^\nu, \hat{\gamma}] + i[\hat{\gamma}, \hat{\Gamma}^\nu],$$

where  $\delta\hat{\Gamma}^\nu \in \mathcal{A}_z^{(dS)}$ . The enveloping algebra-valued connection has infinitely many component fields. Nevertheless, it was shown that all the component fields can be induced from a Lie algebra-valued connection by a Seiberg–Witten map ([2, 3, 5] and [16] for  $SO(n)$  and  $Sp(n)$ ). In this subsection we show that similar constructions could be proposed for nonlinear realizations of de Sitter algebra when the transformation of the connection is considered

$$\delta\hat{\Gamma}^\nu = -i[u^\nu, * \hat{\gamma}] + i[\hat{\gamma}, * \hat{\Gamma}^\nu].$$

For simplicity, we treat in more detail the canonical case with the star product (4). The first term in the variation  $\delta\hat{\Gamma}^\nu$  gives

$$-i[u^\nu, * \hat{\gamma}] = \theta^{\nu\mu} \frac{\partial}{\partial u^\mu} \gamma.$$

Assuming that the variation of  $\hat{\Gamma}^\nu = \theta^{\nu\mu} Q_\mu$  starts with a linear term in  $\theta$  we have

$$\delta\hat{\Gamma}^\nu = \theta^{\nu\mu} \delta Q_\mu, \delta Q_\mu = \frac{\partial}{\partial u^\mu} \gamma + i[\hat{\gamma}, * Q_\mu].$$

We follow the method of calculation from the papers [4, 5] and expand the star product (4) in  $\theta$  but not in  $g_a$  and find to first order in  $\theta$ ,

$$\begin{aligned} \gamma &= \gamma_{\underline{a}}^1 I^{\underline{a}} + \gamma_{\underline{ab}}^1 I^{\underline{a}} I^{\underline{b}} + \dots, \\ Q_\mu &= q_{\mu, \underline{a}}^1 I^{\underline{a}} + q_{\mu, \underline{ab}}^2 I^{\underline{a}} I^{\underline{b}} + \dots \end{aligned} \tag{10}$$

where  $\gamma_{\underline{a}}^1$  and  $q_{\mu,\underline{a}}^1$  are of order zero in  $\theta$  and  $\gamma_{\underline{ab}}^1$  and  $q_{\mu,\underline{ab}}^2$  are of second order in  $\theta$ . The expansion in  $I^b$  leads to an expansion in  $g_a$  of the  $*$ -product because the higher order  $I^b$ -derivatives vanish. For de Sitter case as  $I^b$  we take the generators (6), see commutators (3), with the corresponding de Sitter structure constants  $f_{\underline{a}}^{bc} \simeq f_{\underline{\beta}}^{\underline{\alpha}\underline{\gamma}}$  (in our further identifications with spacetime objects like frames and connections we shall use Greek indices).

The result of calculation of variations of (10), by using  $g_a$  to the order given in (2), is

$$\begin{aligned}\delta q_{\mu,\underline{a}}^1 &= \frac{\partial \gamma_{\underline{a}}^1}{\partial u^\mu} - f_{\underline{a}}^{bc} \gamma_{\underline{b}}^1 q_{\mu,\underline{c}}^1, \\ \delta Q_\tau &= \theta^{\mu\nu} \partial_\mu \gamma_{\underline{a}}^1 \partial_\nu q_{\tau,\underline{b}}^1 I^{\underline{a}} I^{\underline{b}} + \dots, \\ \delta q_{\mu,\underline{ab}}^2 &= \partial_\mu \gamma_{\underline{ab}}^2 - \theta^{\nu\tau} \partial_\nu \gamma_{\underline{a}}^1 \partial_\tau q_{\mu,\underline{b}}^1 - 2f_{\underline{a}}^{bc} \{ \gamma_{\underline{b}}^1 q_{\mu,\underline{cd}}^2 + \gamma_{\underline{bd}}^2 q_{\mu,\underline{c}}^1 \}.\end{aligned}$$

Next we introduce the objects  $\varepsilon$ , taking the values in de Sitter Lie algebra and  $W_\mu$ , being enveloping de Sitter algebra valued,

$$\varepsilon = \gamma_{\underline{a}}^1 I^{\underline{a}} \text{ and } W_\mu = q_{\mu,\underline{ab}}^2 I^{\underline{a}} I^{\underline{b}}$$

with the variation  $\delta W_\mu$  satisfying the equation [4, 5]

$$\delta W_\mu = \partial_\mu (\gamma_{\underline{ab}}^2 I^{\underline{a}} I^{\underline{b}}) - \frac{1}{2} \theta^{\tau\lambda} \{ \partial_\tau \varepsilon, \partial_\lambda q_\mu \} + i[\varepsilon, W_\mu] + i[(\gamma_{\underline{ab}}^2 I^{\underline{a}} I^{\underline{b}}), q_\nu]. \quad (11)$$

The equation (11) has the solution (found in ([4, 2]))

$$\begin{aligned}\gamma_{\underline{ab}}^2 &= \frac{1}{2} \theta^{\nu\mu} (\partial_\nu \gamma_{\underline{a}}^1) q_{\mu,\underline{b}}^1, \\ q_{\mu,\underline{ab}}^2 &= -\frac{1}{2} \theta^{\nu\tau} q_{\nu,\underline{a}}^1 (\partial_\tau q_{\mu,\underline{b}}^1 + R_{\tau\mu,\underline{b}}^1)\end{aligned}$$

where

$$R_{\tau\mu,\underline{b}}^1 = \partial_\tau q_{\mu,\underline{b}}^1 - \partial_\mu q_{\tau,\underline{b}}^1 + f_{\underline{a}}^{ec} q_{\tau,\underline{e}}^1 q_{\mu,\underline{c}}^1$$

could be identified with the coefficients  $\mathcal{R}_{\underline{\beta}\mu\nu}^{\underline{\alpha}}$  of de Sitter nonlinear gauge gravity curvature (see formula (2a) from the Appendix) if in the commutative limit  $q_{\mu,\underline{b}}^1 \simeq \begin{pmatrix} \Gamma_{\underline{\beta}}^{\underline{\alpha}} & l_0^{-1} \chi_{\underline{\beta}}^{\underline{\alpha}} \\ l_0^{-1} \chi_{\underline{\beta}}^{\underline{\alpha}} & 0 \end{pmatrix}$  (see (1a)).

The below presented procedure can be generalized to all the higher powers of  $\theta$  [5].

## 4 Noncommutative Gauge Gravity Covariant Dynamics

### 4.1 First order corrections to gravitational curvature

The constructions from the previous section are summarized by the conclusion that the de Sitter algebra valued object  $\varepsilon = \gamma_{\underline{a}}^1(u) I^{\underline{a}}$  determines all the terms

in the enveloping algebra

$$\gamma = \gamma_{\underline{a}}^1 I^{\underline{a}} + \frac{1}{4} \theta^{\nu\mu} \partial_\nu \gamma_{\underline{a}}^1 q_{\mu,\underline{b}}^1 (I^{\underline{a}} I^{\underline{b}} + I^{\underline{b}} I^{\underline{a}}) + \dots$$

and the gauge transformations are defined by  $\gamma_{\underline{a}}^1(u)$  and  $q_{\mu,\underline{b}}^1(u)$ , when

$$\delta_{\gamma^1} \psi = i\gamma(\gamma^1, q_\mu^1) * \psi.$$

For de Sitter enveloping algebras one holds the general formula for compositions of two transformations

$$\delta_\gamma \delta_\zeta - \delta_\zeta \delta_\gamma = \delta_{i(\zeta * \gamma - \gamma * \zeta)}$$

which holds also for the restricted transformations defined by  $\gamma^1$ ,

$$\delta_{\gamma^1} \delta_{\zeta^1} - \delta_{\zeta^1} \delta_{\gamma^1} = \delta_{i(\zeta^1 * \gamma^1 - \gamma^1 * \zeta^1)}.$$

Applying the formula (4) we computer

$$\begin{aligned} [\gamma, * \zeta] &= i\gamma_{\underline{a}}^1 \zeta_{\underline{b}}^1 f_{\underline{c}}^{ab} I^{\underline{c}} + \frac{i}{2} \theta^{\nu\mu} \{ \partial_\nu (\gamma_{\underline{a}}^1 \zeta_{\underline{b}}^1 f_{\underline{c}}^{ab}) q_{\mu,\underline{c}} \\ &\quad + (\gamma_{\underline{a}}^1 \partial_\nu \zeta_{\underline{b}}^1 - \zeta_{\underline{a}}^1 \partial_\nu \gamma_{\underline{b}}^1) q_{\mu,\underline{b}} f_{\underline{c}}^{ab} + 2\partial_\nu \gamma_{\underline{a}}^1 \partial_\mu \zeta_{\underline{b}}^1 \} I^{\underline{d}} I^{\underline{c}}. \end{aligned}$$

Such commutators could be used for definition of tensors [4]

$$\hat{S}^{\mu\nu} = [\hat{U}^\mu, \hat{U}^\nu] - i\hat{\theta}^{\mu\nu}, \quad (12)$$

where  $\hat{\theta}^{\mu\nu}$  is respectively stated for the canonical, Lie and quantum plane structures. Under the general enveloping algebra one holds the transform

$$\delta \hat{S}^{\mu\nu} = i[\hat{\gamma}, \hat{S}^{\mu\nu}].$$

For instance, the canonical case is characterized by

$$\begin{aligned} S^{\mu\nu} &= i\theta^{\mu\tau} \partial_\tau \Gamma^\nu - i\theta^{\nu\tau} \partial_\tau \Gamma^\mu + \Gamma^\mu * \Gamma^\nu - \Gamma^\nu * \Gamma^\mu \\ &= \theta^{\mu\tau} \theta^{\nu\lambda} \{ \partial_\tau Q_\lambda - \partial_\lambda Q_\tau + Q_\tau * Q_\lambda - Q_\lambda * Q_\tau \}. \end{aligned}$$

By introducing the gravitational gauge strength (curvature)

$$R_{\tau\lambda} = \partial_\tau Q_\lambda - \partial_\lambda Q_\tau + Q_\tau * Q_\lambda - Q_\lambda * Q_\tau, \quad (13)$$

which could be treated as a noncommutative extension of de Sitter nonlinear gauge gravitational curvature (2a), one computers

$$R_{\tau\lambda,\underline{a}} = R_{\tau\lambda,\underline{a}}^1 + \theta^{\mu\nu} \{ R_{\tau\mu,\underline{a}}^1 R_{\lambda\nu,\underline{b}}^1 - \frac{1}{2} q_{\mu,\underline{a}}^1 [(D_\nu R_{\tau\lambda,\underline{b}}^1) + \partial_\nu R_{\tau\lambda,\underline{b}}^1] \} I^{\underline{b}},$$

where the gauge gravitation covariant derivative is introduced,

$$(D_\nu R_{\tau\lambda,\underline{b}}^1) = \partial_\nu R_{\tau\lambda,\underline{b}}^1 + q_{\nu,\underline{c}} R_{\tau\lambda,\underline{d}}^1 f_{\underline{b}}^{cd}.$$

Following the gauge transformation laws for  $\gamma$  and  $q^1$  we find

$$\delta_{\gamma^1} R_{\tau\lambda}^1 = i[\gamma, * R_{\tau\lambda}^1]$$

with the restricted form of  $\gamma$ .

Such formulas were proved in references [5, 2] for usual gauge (nongravitational) fields. Here we reconsidered them for gravitational gauge fields.



## 4.2 Gauge covariant gravitational dynamics

Following the nonlinear realization of de Sitter algebra and the  $*$ -formalism we can formulate a dynamics of noncommutative spaces. Derivatives can be introduced in such a way that one does not obtain new relations for the coordinates. In this case a Leibniz rule can be defined [5] that

$$\hat{\partial}_\mu \hat{u}^\nu = \delta_\mu^\nu + d_{\mu\sigma}^{\nu\tau} \hat{u}^\sigma \hat{\partial}_\tau$$

where the coefficients  $d_{\mu\sigma}^{\nu\tau} = \delta_\sigma^\nu \delta_\mu^\tau$  are chosen to have not new relations when  $\hat{\partial}_\mu$  acts again to the right hand side. In consequence one holds the  $*$ -derivative formulas

$$\begin{aligned} \partial_\tau * f &= \frac{\partial}{\partial u^\tau} f + f * \partial_\tau, \\ [\partial_l, * (f * g)] &= ([\partial_l, * f]) * g + f * ([\partial_l, * g]) \end{aligned}$$

and the Stokes theorem

$$\int [\partial_l, f] = \int d^N u [\partial_l, * f] = \int d^N u \frac{\partial}{\partial u^l} f = 0,$$

where, for the canonical structure, the integral is defined,

$$\int \hat{f} = \int d^N u f(u^1, \dots, u^N).$$

An action can be introduced by using such integrals. For instance, for a tensor of type (12), when

$$\delta \hat{L} = i [\hat{\gamma}, \hat{L}],$$

we can define a gauge invariant action

$$W = \int d^N u \text{Tr} \hat{L}, \quad \delta W = 0,$$

where the trace has to be taken for the group generator.

For the nonlinear de Sitter gauge gravity a proper action is

$$L = \frac{1}{4} R_{\tau\lambda} R^{\tau\lambda},$$

where  $R_{\tau\lambda}$  is defined by (13) (in the commutative limit we shall obtain the connection (1a)). In this case the dynamic of noncommutative space is entirely formulated in the framework of quantum field theory of gauge fields. The method works for matter fields as well to restrictions to the general relativity theory (see references [12, 11] and the Appendix).

## Appendix: De Sitter Nonlinear Gauge Gravity and General Relativity

Let us consider the de Sitter nonlinear gauge gravitational connection (9) rewritten in the form

$$\Gamma = \begin{pmatrix} \Gamma^\alpha_\beta & l_0^{-1} \chi^\alpha \\ l_0^{-1} \chi_\beta & 0 \end{pmatrix} \quad (1a)$$

where

$$\begin{aligned}\Gamma^{\underline{\alpha}}{}_{\underline{\beta}} &= \Gamma^{\underline{\alpha}}{}_{\underline{\beta}\mu} \delta u^{\mu}, \\ \Gamma^{\underline{\alpha}}{}_{\underline{\beta}\mu} &= \chi^{\underline{\alpha}}{}_{\alpha} \chi^{\underline{\beta}}{}_{\beta} \Gamma^{\alpha}{}_{\beta\gamma} + \chi^{\underline{\alpha}}{}_{\alpha} \delta_{\mu} \chi^{\alpha}{}_{\underline{\beta}}, \\ \chi^{\underline{\alpha}} &= \chi^{\underline{\alpha}}{}_{\mu} \delta u^{\mu},\end{aligned}$$

and

$$G_{\alpha\beta} = \chi^{\underline{\alpha}}{}_{\alpha} \chi^{\underline{\beta}}{}_{\beta} \eta_{\underline{\alpha}\underline{\beta}},$$

$\eta_{\underline{\alpha}\underline{\beta}} = (1, -1, \dots, -1)$  and  $l_0$  is a dimensional constant.  
The curvature of (1a),

$$\mathcal{R}^{(\Gamma)} = d\Gamma + \Gamma \wedge \Gamma,$$

can be written

$$\mathcal{R}^{(\Gamma)} = \begin{pmatrix} \mathcal{R}^{\underline{\alpha}}{}_{\underline{\beta}} + l_0^{-1} \pi_{\underline{\beta}}^{\underline{\alpha}} & l_0^{-1} T^{\underline{\alpha}} \\ l_0^{-1} T^{\underline{\beta}} & 0 \end{pmatrix}, \quad (2a)$$

where

$$\pi_{\underline{\beta}}^{\underline{\alpha}} = \chi^{\underline{\alpha}} \wedge \chi_{\underline{\beta}}, \quad \mathcal{R}^{\underline{\alpha}}{}_{\underline{\beta}} = \frac{1}{2} \mathcal{R}^{\underline{\alpha}}{}_{\underline{\beta}\mu\nu} \delta u^{\mu} \wedge \delta u^{\nu},$$

and

$$\mathcal{R}^{\underline{\alpha}}{}_{\underline{\beta}\mu\nu} = \chi_{\underline{\beta}}{}^{\beta} \chi_{\alpha}{}^{\underline{\alpha}} R^{\alpha}{}_{\beta\mu\nu}.$$

with  $R^{\alpha}{}_{\beta\mu\nu}$  being the metric-affine (for Einstein-Cartan-Weyl spaces), or (pseudo) Riemannian curvature. The de Sitter gauge group is semisimple and we are able to construct a variational gauge gravitational theory with the Lagrangian

$$L = L_{(G)} + L_{(m)}$$

where the gauge gravitational Lagrangian is defined

$$L_{(G)} = \frac{1}{4\pi} \text{Tr} \left( \mathcal{R}^{(\Gamma)} \wedge *_G \mathcal{R}^{(\Gamma)} \right) = \mathcal{L}_{(G)} |G|^{1/2} \delta^4 u,$$

with

$$\mathcal{L}_{(G)} = \frac{1}{2l^2} T^{\underline{\alpha}}{}_{\mu\nu} T^{\mu\nu}{}_{\underline{\alpha}} + \frac{1}{8\lambda} \mathcal{R}^{\underline{\alpha}}{}_{\underline{\beta}\mu\nu} \mathcal{R}^{\underline{\beta}}{}_{\underline{\alpha}}{}^{\mu\nu} - \frac{1}{l^2} \left( \overleftarrow{R}(\Gamma) - 2\lambda_1 \right), \quad (3a)$$

$\delta^4 u$  being the volume element,  $T^{\underline{\alpha}}{}_{\mu\nu} = \chi^{\underline{\alpha}}{}_{\alpha} T^{\alpha}{}_{\mu\nu}$  (the gravitational constant  $l^2$  in (3a) satisfies the relations  $l^2 = 2l_0^2 \lambda$ ,  $\lambda_1 = -3/l_0$ ),  $\text{Tr}$  denotes the trace on  $\underline{\alpha}, \underline{\beta}$  indices, and the matter field Lagrangian is defined

$$L_{(m)} = -1 \frac{1}{2} \text{Tr} \left( \Gamma \wedge *_G \mathcal{I} \right) = \mathcal{L}_{(m)} |G|^{1/2} \delta^n u,$$

where

$$\mathcal{L}_{(m)} = \frac{1}{2} \Gamma^{\underline{\alpha}}{}_{\underline{\beta}\mu} S^{\underline{\beta}}{}_{\alpha}{}^{\mu} - t^{\mu}{}_{\underline{\alpha}} l^{\underline{\alpha}}{}_{\mu}. \quad (4a)$$

The matter field source  $\mathcal{J}$  is obtained as a variational derivation of  $\mathcal{L}_{(m)}$  on  $\Gamma$  and is parametrized as

$$\mathcal{J} = \begin{pmatrix} S^\alpha_{\underline{\beta}} & -l_0 t^\alpha_{\underline{\beta}} \\ -l_0 t^\alpha_{\underline{\beta}} & 0 \end{pmatrix} \quad (5a)$$

with  $t^\alpha = t^\alpha_{\underline{\mu}} \delta u^\mu$  and  $S^\alpha_{\underline{\beta}} = S^\alpha_{\underline{\beta}\mu} \delta u^\mu$  being respectively the canonical tensors of energy–momentum and spin density.

Varying the action

$$S = \int \delta^4 u \left( \mathcal{L}_{(G)} + \mathcal{L}_{(m)} \right)$$

on the  $\Gamma$ –variables (1a), we obtain the gauge–gravitational field equations:

$$d \left( * \mathcal{R}^{(\Gamma)} \right) + \Gamma \wedge \left( * \mathcal{R}^{(\Gamma)} \right) - \left( * \mathcal{R}^{(\Gamma)} \right) \wedge \Gamma = -\lambda (* \mathcal{J}), \quad (6a)$$

where the Hodge operator  $*$  is used.

Specifying the variations on  $\Gamma^\alpha_{\underline{\beta}}$  and  $\chi$ –variables, we rewrite (6a)

$$\begin{aligned} \widehat{\mathcal{D}} \left( * \mathcal{R}^{(\Gamma)} \right) + \frac{2\lambda}{l^2} \left( \widehat{\mathcal{D}} (* \pi) + \chi \wedge (* T^T) - (* T) \wedge \chi^T \right) &= -\lambda (* S), \\ \widehat{\mathcal{D}} (* T) - \left( * \mathcal{R}^{(\Gamma)} \right) \wedge \chi - \frac{2\lambda}{l^2} (* \pi) \wedge \chi &= \frac{l^2}{2} \left( * t + \frac{1}{\lambda} * \tau \right), \end{aligned}$$

where

$$\begin{aligned} T^t &= \{ T_{\underline{\alpha}} = \eta_{\underline{\alpha}\underline{\beta}} T^{\underline{\beta}}, T^{\underline{\beta}} = \frac{1}{2} T^{\underline{\beta}}_{\mu\nu} \delta u^\mu \wedge \delta u^\nu \}, \\ \chi^T &= \{ \chi_{\underline{\alpha}} = \eta_{\underline{\alpha}\underline{\beta}} \chi^{\underline{\beta}}, \chi^{\underline{\beta}} = \chi^{\widehat{\beta}}_{\mu} \delta u^\mu \}, \quad \widehat{\mathcal{D}} = d + \widehat{\Gamma}, \end{aligned}$$

( $\widehat{\Gamma}$  acts as  $\Gamma^\alpha_{\underline{\beta}\mu}$  on indices  $\underline{\gamma}, \underline{\delta}, \dots$  and as  $\Gamma^\alpha_{\beta\mu}$  on indices  $\gamma, \delta, \dots$ ). The value  $\tau$  defines the energy–momentum tensor of the gauge gravitational field  $\widehat{\Gamma}$ :

$$\tau_{\mu\nu} (\widehat{\Gamma}) = \frac{1}{2} Tr \left( \mathcal{R}_{\mu\alpha} \mathcal{R}^\alpha_{\nu} - \frac{1}{4} \mathcal{R}_{\alpha\beta} \mathcal{R}^{\alpha\beta} G_{\mu\nu} \right).$$

Equations (6a) (or equivalently (7a)) make up the complete system of variational field equations for nonlinear de Sitter gauge gravity.

We note that we can obtain a nonvariational Poincare gauge gravitational theory if we consider the contraction of the gauge potential (1a) to a potential with values in the Poincare Lie algebra

$$\Gamma = \begin{pmatrix} \Gamma^{\widehat{\alpha}}_{\widehat{\beta}} & l_0^{-1} \chi^{\widehat{\alpha}}_{\widehat{\beta}} \\ l_0^{-1} \chi^{\widehat{\alpha}}_{\widehat{\beta}} & 0 \end{pmatrix} \rightarrow \Gamma = \begin{pmatrix} \Gamma^{\widehat{\alpha}}_{\widehat{\beta}} & l_0^{-1} \chi^{\widehat{\alpha}}_{\widehat{\beta}} \\ 0 & 0 \end{pmatrix}. \quad (7a)$$

A similar gauge potential was considered in the formalism of linear and affine frame bundles on curved spacetimes by Popov and Dikhhin [11]. They treated (7a) as the Cartan connection form for affine gauge like gravity and by using ‘pure’ geometric methods proved that the Yang–Mills equations of their

theory are equivalent, after projection on the base, to the Einstein equations. The main conclusion for a such approach to Einstein gravity is that this theory admits an equivalent formulation as gauge model but with nonsemisimple structural gauge groups. In order to have a variational theory on the total bundle space it is necessary to introduce an auxiliary bilinear form on the typical fiber, instead of degenerated Killing form; the coefficients of auxiliary form disappear after pojection on the base. An alternative variant is to consider a gauge gravitational theory when the gauge group was minimally extended to the de Sitter one with nondegenerated Killing form. The nonlinear realizations have to be introduced if we wont to consider in a common fashion both the frame (tetradic) and connection components included as the coefficients of the potential (1a).

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